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NATURAL FREQUENCIES OF MULTI-SPAN CIRCULAR CURVED FRAMES

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Abstract—The general dynamic slope–deflection equations for circular curved members of constant section have been derived for the determination of natural frequencies of frame structures. An example of a two-span curved frame is given to illustrate the application of the derived equations and to show the effect of the central angle of the arc upon the natural frequencies of the frame.

1. INTRODUCTION

VIBRATIONS of curved beams have been studied by many investigators. Den Hartog [1] applied the Rayleigh–Ritz method to find the lowest natural frequency of circular arcs vibrating in the plane of initial curvature of the arc. Volterra and Morell [2] extended Den Hartog's work to include arcs having center lines in the form of cycloids, catenaries or parabolas. The first detailed paper concerning the vibration of ring segments was published by Waltking [3] who obtained the exact solution for the free vibration of a pinned–pinned circular arc. Morley [4] solved the problem of flexural vibrations of a cut thin ring exactly and presented the first ten modes of symmetrical and anti-symmetrical vibrations. The inextensional vibrations of an incomplete circular ring with additional terms to represent damping effects were studied by Archer [5]. Using the Rayleigh–Ritz technique in conjunction with Lagrangian multipliers, Nelson [6] made an analytical study of the in-plane vibration of a simply supported circular ring segment. He obtained frequency equations in the form of infinite series for inextensional and extensional in both symmetrical and anti-symmetrical mode shapes.

All these works mentioned above are simple curved beams with various boundary conditions. No investigations, however, have been made for curved frames. The purpose of this paper is, therefore, to present a general method for analyzing circular curved frames, single or continuous. Similar to those in the statical case [7], the general dynamic slope–deflection equations for circular curved members in terms of rotation, vertical and horizontal displacements, have been derived. The use of the derived equations is then illustrated by the determination of natural frequencies of a circular curved frame.

2. BASIC DIFFERENTIAL EQUATION AND ITS SOLUTION

Consider the in-plane, small vibration of a circular curved element as shown in Fig. 1. The equations of motions in radial and tangential directions and the moment equation are

$$\frac{\partial Q}{\partial \theta}+N = mr\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial N}{\partial \theta}-Q = mr\frac{\partial^2 w}{\partial t^2} \tag{2}$$

$$-\frac{\partial M}{\partial \theta}+Qr = 0 \tag{3}$$

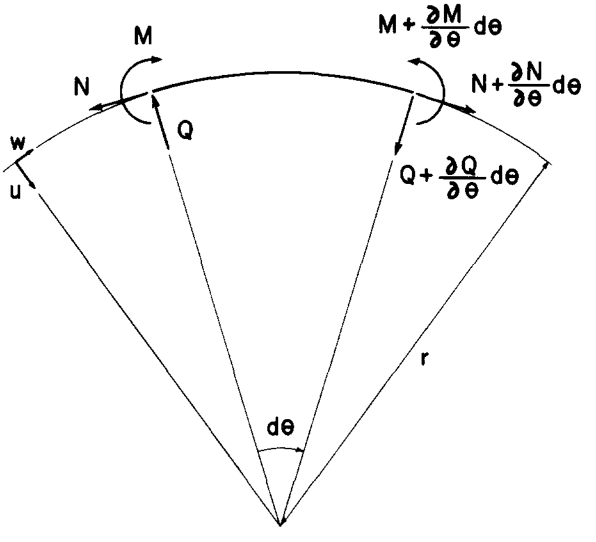

FIG. 1. Element of member subjected to forces and moments.

where Q is the shear force, N the normal force, M the bending moment, m the mass per unit length of member, r the radius of circular arc, θ the angular coordinate, u the inward radial displacement, w the tangential displacement in the sense of increasing θ and t the time. For inextensional vibration, the displacements must satisfy the condition

$$u = \frac{\partial w}{\partial \theta}. \tag{4}$$

The relation between the moment and the change in curvature [8] takes the form of

$$M = -\frac{EI}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}+u\right) \tag{5}$$

where E is the modulus of elasticity, I the moment of inertia of cross section and M, r, u and θ as defined previously.

From equations (1) and (3)–(5) we obtain

$$Q = -\frac{EI}{r^3}\left(\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}\right) \tag{6}$$

$$N = mr\frac{\partial^3 w}{\partial t^2\partial\theta}+\frac{EI}{r^3}\left(\frac{\partial^5 w}{\partial\theta^5}+\frac{\partial^3 w}{\partial\theta^3}\right). \tag{7}$$

Substituting equations (6) and (7) in equation (2) yields a sixth-order differential equation in w as follows:

$$\frac{EI}{r^4}\left(\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}\right) = m\left(\frac{\partial^2 w}{\partial t^2}-\frac{\partial^4 w}{\partial t^2\partial\theta^2}\right). \tag{8}$$

Assuming that the curved member is excited harmonically with a frequency p and letting

$$w(\theta, t) = W(\theta)\,.\,e^{ipt} \tag{9}$$

where $W(\theta)$ is the tangential modal function and $i = \sqrt{(-1)}$, substituting equation (9) in (8) and omitting the common term e^{ipt}, the following equation is obtained:

$$W^{VI}+2W^{IV}+(1-\lambda)W''+\lambda W = 0 \tag{10}$$

where

$$\lambda = \frac{mr^4p^2}{EI} \tag{11}$$

and the primes for W represent differentiation with respect to θ.

Equation (10) is a linear differential equation with constant coefficients so the standard form of solution for $W(\theta)$ is

$$W(\theta) = \sum_{n=1}^{6} a_n\, e^{y_n\theta} \tag{12}$$

where y_n are the roots of the auxiliary equation. The roots are of three types depending upon the value of λ.

Case 1: $0 < \lambda < 0{\cdot}113407$

The roots are of the form

$$y_{1,2} = \pm\sigma_1 i, \qquad y_{3,4} = \pm\sigma_2 i, \qquad y_{5,6} = \pm\sigma_3 i.$$

Case 2: $0{\cdot}113407 < \lambda < 17{\cdot}6366$

The six roots are

$$y_{1,2} = \pm\sigma_1 i, \qquad y_{3,4} = \pm(\nu+\mu i), \qquad y_{5,6} = \pm(\nu-\mu i).$$

Case 3: $17{\cdot}6366 < \lambda < \infty$

The roots in this case are

$$y_{1,2} = \pm\sigma_1 i, \qquad y_{3,4} = \pm\sigma_2, \qquad y_{5,6} = \pm\sigma_3.$$

The modal functions for the three cases can be written as

$$W(\theta) = \mathbf{D}(\theta)\mathbf{X} \tag{13}$$

where $\mathbf{D}(\theta)$, a row matrix, is given in the Appendix for the three cases and

$$\mathbf{X} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}.$$

3. GENERAL DYNAMIC SLOPE–DEFLECTION EQUATIONS FOR CIRCULAR CURVED MEMBER OF CONSTANT SECTION

Figure 2 shows a circular curved member of constant cross section subjected to harmonic displacements, linear and rotational, at the two ends A and B.

Consider first the rotation at A with B being fixed. The boundary conditions due to θ_a acting only are

$$\begin{aligned} W(0) &= 0 & W(\alpha) &= 0 \\ U(0) &= 0 & U(\alpha) &= 0 \\ U'(0)+W(0) &= r\theta_a & U'(\alpha)+W(\alpha) &= 0. \end{aligned} \tag{14}$$

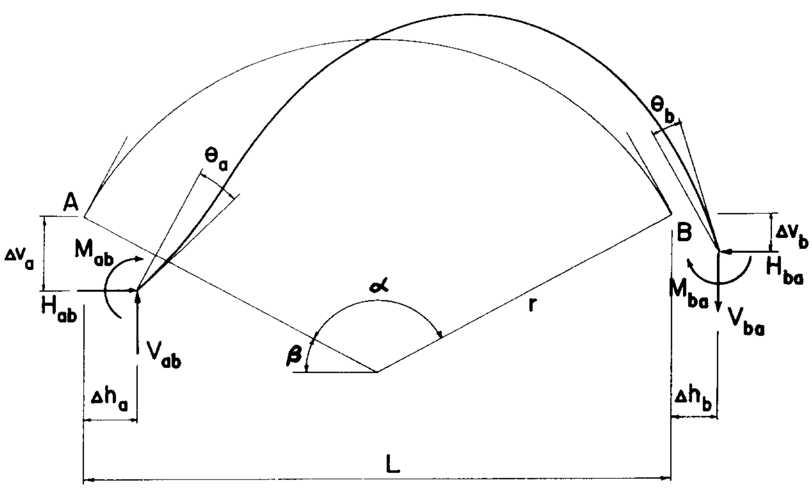

FIG. 2. Positive displacements, forces and moments with common factor e^{ipt} omitted.

The substitution of equation (13) into equations (14) will yield the solution of the unknown coefficients in the following matrix form:

$$\mathbf{X} = r\theta_a \mathbf{A}^{-1}\mathbf{B} \tag{15}$$

where $\mathbf{A}^{-1}$ is the inverse of matrix $\mathbf{A}$ which is a coefficient matrix given in the Appendix and

$$\mathbf{B} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substituting equation (15) in (13) yields

$$W_1(\theta) = r\,.\,\theta_a\,.\,\mathbf{D}(\theta)\,.\,\mathbf{A}^{-1}\,.\,\mathbf{B}. \tag{16}$$

Using equations (4), (5), (14) and (16) the moments at A and B can be obtained as follows:

$$M_{ab1} = \frac{EI}{L} C_{1a}\theta_a \tag{17}$$

$$M_{ba1} = -\frac{EI}{L} C_{1b}\theta_a \tag{18}$$

where

$$C_{1a} = -2\sin\left(\frac{\alpha}{2}\right).\,\mathbf{D}'''(0)\,.\,\mathbf{A}^{-1}\,.\,\mathbf{B} \tag{19}$$

$$C_{1b} = 2\sin\left(\frac{\alpha}{2}\right).\,\mathbf{D}'''(\alpha)\,.\,\mathbf{A}^{-1}\,.\,\mathbf{B} \tag{20}$$

$$L = 2r\sin\left(\frac{\alpha}{2}\right). \tag{21}$$

Consider next a vertical displacement Δv_a at A while B is still fixed. The boundary conditions are

$$\begin{aligned} U'(0)+W(0) &= 0 & W(\alpha) &= 0 \\ U(0)\,.\cos\beta + W(0)\,.\sin\beta &= 0 & U(\alpha) &= 0 \\ U(0)\,.\sin\beta - W(0)\,.\cos\beta &= \Delta v_a & U'(\alpha)+W(\alpha) &= 0. \end{aligned} \tag{22}$$

Using the relations of equations (22), equation (13) gives the following tangential displacement

$$W_2(\theta) = \Delta v_a\,.\,\mathbf{D}(\theta)\,.\,\mathbf{V}^{-1}\,.\,\mathbf{B} \tag{23}$$

where $\mathbf{V}$ is a coefficient matrix which is shown in the Appendix, and $\mathbf{D}(\theta)$ and $\mathbf{B}$ as defined previously.

The moments at both ends can be obtained by using equations (4), (5), (22) and (23). The results are

$$M_{ab2} = \frac{EI}{L^2} C_{2a} \,.\, \Delta v_a \tag{24}$$

$$M_{ba2} = -\frac{EI}{L^2} C_{2b} \,.\, \Delta v_b \tag{25}$$

where

$$C_{2a} = -4 \sin^2\left(\frac{\alpha}{2}\right) . \, [\mathbf{D}'''(0) + \mathbf{D}'(0)] \,.\, \mathbf{V}^{-1} \,.\, \mathbf{B} \tag{26}$$

$$C_{2b} = 4 \sin^2\left(\frac{\alpha}{2}\right) . \, \mathbf{D}'''(\alpha) \,.\, \mathbf{V}^{-1} \,.\, \mathbf{B}. \tag{27}$$

Finally consider a horizontal displacement at A with B being fixed. The boundary conditions in this case are

$$\begin{aligned} U'(0) + W(0) &= 0 & W(\alpha) &= 0 \\ U(0) \,.\, \sin\beta - W(0) \,.\, \cos\beta &= 0 & U(\alpha) &= 0 \\ U(0) \,.\, \cos\beta + W(0) \,.\, \sin\beta &= \Delta h_a & U'(\alpha) + W(\alpha) &= 0. \end{aligned} \tag{28}$$

Similar to the previous cases and from equation (13) we have

$$W_3(\theta) = \Delta h_a \,.\, \mathbf{D}(\theta) \,.\, \mathbf{H}^{-1} \,.\, \mathbf{B} \tag{29}$$

where **H** is given in the Appendix.

The moments at A and B for this case are

$$M_{ab3} = \frac{EI}{L^2} C_{3a} \,.\, \Delta h_a \tag{30}$$

$$M_{ba3} = -\frac{EI}{L^2} C_{3b} \,.\, \Delta h_a \tag{31}$$

where

$$C_{3a} = -4 \sin^2\left(\frac{\alpha}{2}\right) . \, [\mathbf{D}'''(0) + \mathbf{D}'(0)] \,.\, \mathbf{H}^{-1} \,.\, \mathbf{B} \tag{32}$$

$$C_{3b} = 4 \sin^2\left(\frac{\alpha}{2}\right) . \, \mathbf{D}'''(\alpha) \,.\, \mathbf{H}^{-1} \,.\, \mathbf{B}. \tag{33}$$

The moments at both ends of the member due to θ_b, Δv_b and Δh_b at B can be obtained in the same manner.

Having considered the effects due to these displacements acting separately, the general dynamic slope–deflection equations for moments can now be obtained by combining the results due to these effects and they are

$$M_{ab} = \frac{EI}{L}\left(C_{1a}\theta_a + C_{1b}\theta_b + C_{2a}\frac{\Delta v_a}{L} - C_{2b}\frac{\Delta v_b}{L} + C_{3a}\frac{\Delta h_a}{L} + C_{3b}\frac{\Delta h_b}{L}\right) \tag{34}$$

$$M_{ba} = \frac{EI}{L}\left(C_{1b}\theta_a + C_{1a}\theta_b + C_{2b}\frac{\Delta v_a}{L} - C_{2a}\frac{\Delta v_b}{L} + C_{3b}\frac{\Delta h_a}{L} + C_{3a}\frac{\Delta h_b}{L}\right). \tag{35}$$

The slope–deflection equations for vertical and horizontal thrusts can be derived in the following manner.

From equations (6) and (7) the shear and normal forces at any section are

$$\bar{Q}(\theta) = -\frac{EI}{r^3}[W^{IV}(\theta) + W''(\theta)] \tag{36}$$

$$\bar{N}(\theta) = \frac{EI}{r^3}[W^{V}(\theta) + W'''(\theta) - \lambda \cdot W'(\theta)]. \tag{37}$$

The vertical and horizontal thrusts at A and B are

$$V_{ab} = \bar{Q}(0)\,.\sin\beta - \bar{N}(0)\,.\cos\beta$$
$$H_{ab} = -\bar{Q}(0)\,.\cos\beta - \bar{N}(0)\,.\sin\beta$$
$$V_{ba} = \bar{Q}(\alpha)\,.\sin(\alpha+\beta) - \bar{N}(\alpha)\,.\cos(\alpha+\beta)$$
$$H_{ba} = -\bar{Q}(\alpha)\,.\cos(\alpha+\beta) - \bar{N}(\alpha)\,.\sin(\alpha+\beta).$$

Substitution of equations (36) and (37) into the above four equations yields

$$V_{ab} = \frac{8EI}{L^3}\sin^3\left(\frac{\alpha}{2}\right)\{-[W^{IV}(0) + W''(0)]\sin\beta - [W^{V}(0) + W'''(0) - \lambda W'(0)]\cos\beta\} \tag{38}$$

$$H_{ab} = \frac{8EI}{L^3}\sin^3\left(\frac{\alpha}{2}\right)\{[W^{IV}(0) + W''(0)]\cos\beta - [W^{V}(0) + W'''(0) - \lambda W'(0)]\sin\beta\} \tag{39}$$

$$V_{ba} = \frac{8EI}{L^3}\sin^3\left(\frac{\alpha}{2}\right)\{-[W^{IV}(\alpha) + W''(\alpha)]\sin(\alpha+\beta) - [W^{V}(\alpha) + W'''(\alpha) - \lambda W'(\alpha)]\cos(\alpha+\beta)\} \tag{40}$$

$$H_{ba} = \frac{8EI}{L^3}\sin^3\left(\frac{\alpha}{2}\right)\{[W^{IV}(\alpha) + W''(\alpha)]\cos(\alpha+\beta) - [W^{V}(\alpha) + W'''(\alpha) - \lambda W'(\alpha)]\sin(\alpha+\beta)\}. \tag{41}$$

The vertical thrust due to a rotation at A can be obtained by substituting equation (16) into (38). Thus

$$V_{ab1} = \frac{EI}{L^2}F_{1a}\theta_a \tag{42}$$

in which

$$F_{1a} = -4\sin^2\left(\frac{\alpha}{2}\right).\,\mathbf{J}(0,\beta)\,.\,\mathbf{A}^{-1}\,.\,\mathbf{B} \tag{43}$$

and

$$\mathbf{J}(\theta,\phi) = [\mathbf{D}^{IV}(\theta) + \mathbf{D}''(\theta)]\sin\phi + [\mathbf{D}^{V}(\theta) + \mathbf{D}'''(\theta) - \lambda\mathbf{D}'(\theta)]\cos\phi \tag{44}$$

where ϕ is a dummy vector.

Similarly, from equations (23), (29) and (38), the thrusts due to vertical and horizontal displacements at A are, respectively,

$$V_{ab2} = \frac{EI}{L^3} F_{2a} \,.\, \Delta v_a \tag{45}$$

$$V_{ab3} = \frac{EI}{L^3} F_{3a} \,.\, \Delta h_a \tag{46}$$

where

$$F_{2a} = -8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{J}(0, \beta) \cdot \mathbf{V}^{-1} . \mathbf{B} \tag{47}$$

$$F_{3a} = -8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{J}(0, \beta) . \mathbf{H}^{-1} . \mathbf{B}. \tag{48}$$

The horizontal thrusts due to rotation, vertical and horizontal displacements at A can be obtained in the same manner. Substitution of equations (16), (23) and (29), respectively, into equation (39) yields

$$H_{ab1} = \frac{EI}{L^2} G_{1a} \,.\, \theta_a \tag{49}$$

$$H_{ab2} = \frac{EI}{L^3} G_{2a} \,.\, \Delta v_a \tag{50}$$

$$H_{ab3} = \frac{EI}{L^3} G_{3a} \,.\, \Delta h_a \tag{51}$$

in which

$$G_{1a} = 4 \sin^2\left(\frac{\alpha}{2}\right) . \mathbf{K}(0, \beta) . \mathbf{A}^{-1} . \mathbf{B} \tag{52}$$

$$G_{2a} = 8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{K}(0, \beta) . \mathbf{V}^{-1} . \mathbf{B} \tag{53}$$

$$G_{3a} = 8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{K}(0, \beta) . \mathbf{H}^{-1} . \mathbf{B} \tag{54}$$

and

$$\mathbf{K}(\theta, \phi) = [\mathbf{D}^{IV}(\theta) + \mathbf{D}''(\theta)] \cos \phi - [\mathbf{D}^{V}(\theta) + \mathbf{D}'''(\theta) - \lambda \mathbf{D}'(\theta)] \sin \phi. \tag{55}$$

Following the same procedure as before, the vertical and horizontal thrusts at B due to θ_a, Δv_a and Δh_a are

$$V_{ba1} = \frac{EI}{L^2} F_{1b} \,.\, \theta_a \tag{56}$$

$$V_{ba2} = \frac{EI}{L^3} F_{2b} \,.\, \Delta v_a \tag{57}$$

$$V_{ba3} = \frac{EI}{L^3} F_{3b} \,.\, \Delta h_a \tag{58}$$

$$H_{ba1} = \frac{EI}{L^2} G_{1b} \,.\, \theta_a \tag{59}$$

$$H_{ba2} = \frac{EI}{L^3} G_{2b} \,.\, \Delta v_a \tag{60}$$

$$H_{ba3} = \frac{EI}{L^3} G_{3b} \,.\, \Delta h_a \tag{61}$$

where

$$F_{1b} = -4 \sin^2\left(\frac{\alpha}{2}\right) . \mathbf{J}(\alpha, \alpha+\beta) \,.\, \mathbf{A}^{-1} \,.\, \mathbf{B} \tag{62}$$

$$F_{2b} = -8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{J}(\alpha, \alpha+\beta) \,.\, \mathbf{V}^{-1} \,.\, \mathbf{B} \tag{63}$$

$$F_{3b} = -8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{J}(\alpha, \alpha+\beta) \,.\, \mathbf{H}^{-1} \,.\, \mathbf{B} \tag{64}$$

$$G_{1b} = 4 \sin^2\left(\frac{\alpha}{2}\right) . \mathbf{K}(\alpha, \alpha+\beta) \,.\, \mathbf{A}^{-1} \,.\, \mathbf{B} \tag{65}$$

$$G_{2b} = 8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{K}(\alpha, \alpha+\beta) \,.\, \mathbf{V}^{-1} \,.\, \mathbf{B} \tag{66}$$

$$G_{3b} = 8 \sin^3\left(\frac{\alpha}{2}\right) . \mathbf{K}(\alpha, \alpha+\beta) \,.\, \mathbf{H}^{-1} \,.\, \mathbf{B}. \tag{67}$$

The thrusts at both ends due to θ_b, Δv_b and Δh_b can be obtained again in the same way.

Superimposing the effects due to the displacements at both ends of the member we obtain the following slope–deflection equations for thrusts:

$$V_{ab} = \frac{EI}{L^2}\left(F_{1a}\theta_a + F_{1b}\theta_b + F_{2a}\frac{\Delta v_a}{L} - F_{2b}\frac{\Delta v_b}{L} + F_{3a}\frac{\Delta h_a}{L} + F_{3b}\frac{\Delta h_b}{L} \right) \tag{68}$$

$$V_{ba} = \frac{EI}{L^2}\left(F_{1b}\theta_a + F_{1a}\theta_b + F_{2b}\frac{\Delta v_a}{L} - F_{2a}\frac{\Delta v_b}{L} + F_{3b}\frac{\Delta h_a}{L} + F_{3a}\frac{\Delta h_b}{L} \right) \tag{69}$$

$$H_{ab} = \frac{EI}{L^2}\left(G_{1a}\theta_a - G_{1b}\theta_b + G_{2a}\frac{\Delta v_a}{L} + G_{2b}\frac{\Delta v_b}{L} + G_{3a}\frac{\Delta h_a}{L} - G_{3b}\frac{\Delta h_b}{L} \right) \tag{70}$$

$$H_{ba} = \frac{EI}{L^2}\left(G_{1b}\theta_a - G_{1a}\theta_b + G_{2b}\frac{\Delta v_a}{L} + G_{2a}\frac{\Delta v_b}{L} + G_{3b}\frac{\Delta h_a}{L} - G_{3a}\frac{\Delta h_b}{L} \right). \tag{71}$$

The general dynamic slope–deflection equations for moments and thrusts have been derived and they are given in equations (34), (35) and (68)–(71). The coefficients appearing in these equations are functions of α and λ, and can be computed with the aid of a digital computer for different values of α; and C, a frequency constant for curved member, is given by

$$p = C\sqrt{\left(\frac{EI}{mL^4}\right)} \tag{72}$$

where

$$C = 4\sin^2\left(\frac{\alpha}{2}\right) . \sqrt{\lambda}. \tag{73}$$

4. EXAMPLE

A symmetrical circular curved frame of constant cross-section undergoing horizontal vibrations as shown in Fig. 3 is analyzed for natural frequencies. The conditions of dynamic equilibrium at joint B give

$$M_{ba}+M_{bc}+M_{bd} = 0 \tag{74}$$

$$-H_{ba}+H_{bc}+H_{bd} = 0. \tag{75}$$

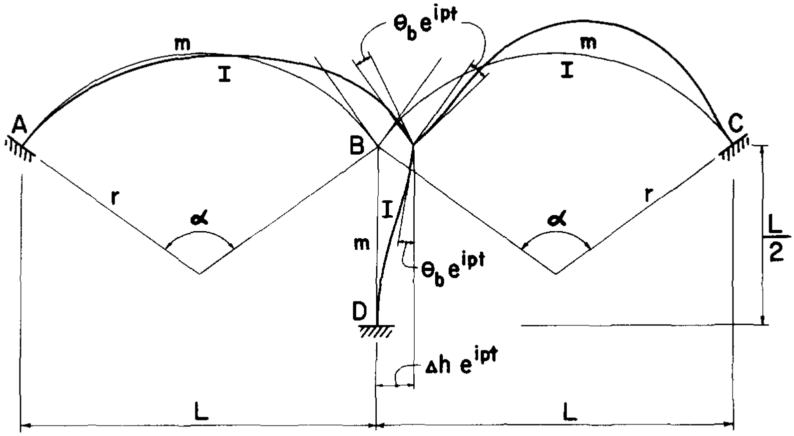

FIG. 3. Frame undergoing horizontal vibrations.

The dynamic slope–deflection equations for BA and BC can be written as

$$M_{ba} = \frac{EI}{L}\left(C_{1a}\theta_b + C_{3a}\frac{\Delta h}{L}\right) \tag{76}$$

$$M_{bc} = \frac{EI}{L}\left(C_{1a}\theta_b + C_{3a}\frac{\Delta h}{L}\right) \tag{77}$$

$$H_{ba} = \frac{EI}{L^2}\left(-G_{1a}\theta_b - G_{3a}\frac{\Delta h}{L}\right) \tag{78}$$

$$H_{bc} = \frac{EI}{L^2}\left(G_{1a}\theta_b + G_{3a}\frac{\Delta h}{L}\right). \tag{79}$$

For column BD where the effect of normal forces is small and can be neglected, the moment and shear take the forms [9]

$$M_{bd} = \frac{2EI}{L}\left[\bar{c}(\bar{\lambda})\theta_b - \bar{t}(\bar{\lambda})\frac{2\Delta h}{L}\right] \tag{80}$$

$$H_{bd} = -\frac{4EI}{L^2}\left[\bar{t}(\bar{\lambda})\theta_b - \bar{m}(\bar{\lambda})\frac{2\Delta h}{L}\right] \tag{81}$$

where

$$\bar{c}(\bar{\lambda}) = \bar{\lambda}\frac{\cosh\bar{\lambda}\sin\bar{\lambda} - \sinh\bar{\lambda}\cos\bar{\lambda}}{1-\cosh\bar{\lambda}\cos\bar{\lambda}} \tag{82}$$

$$\bar{t}(\bar{\lambda}) = \bar{\lambda}^2\frac{\sinh\bar{\lambda}\sin\bar{\lambda}}{1-\cosh\bar{\lambda}\cos\bar{\lambda}} \tag{83}$$

$$\bar{m}(\bar{\lambda}) = \bar{\lambda}^3\frac{\sinh\bar{\lambda}\cos\bar{\lambda} + \cosh\bar{\lambda}\sin\bar{\lambda}}{1-\cosh\bar{\lambda}\cos\bar{\lambda}} \tag{84}$$

and

$$\bar{\lambda} = \frac{L}{2}\sqrt[4]{\left(\frac{mp^2}{EI}\right)}. \tag{85}$$

The relation between $\bar{\lambda}$ and λ is given by

$$\bar{\lambda} = \sin\left(\frac{\alpha}{2}\right) \cdot \sqrt[4]{\lambda}. \tag{86}$$

Substituting equations (76)–(81) in equations (74) and (75) leads to the following frequency equation:

$$\begin{vmatrix} C_{1a}+\bar{c}(\bar{\lambda}) & C_{3a}-2\bar{t}(\bar{\lambda}) \\ G_{1a}-2\bar{t}(\bar{\lambda}) & G_{3a}+4\bar{m}(\bar{\lambda}) \end{vmatrix} = 0. \tag{87}$$

Equation (87), in fact, has only one unknown λ which can be solved by the method of false position on the IBM 360 computer. The results of C vs. α for the first five modes with α varying from 20 to 180° are shown in Fig. 4.

5. CONCLUSIONS

The dynamic slope–deflection equations for circular curved members of constant cross section have been presented in this paper for the determination of the natural frequencies of frame structures. The application of the derived equations has been illustrated in the example of a two-span curved frame undergoing natural horizontal vibrations.

The results given in Fig. 4 show that the natural frequency of the curved frame decreases as the central angle of the arc increases. This effect becomes significant for higher modes. The proposed method can also be extended to the analysis of curved frames subjected to forced vibrations.

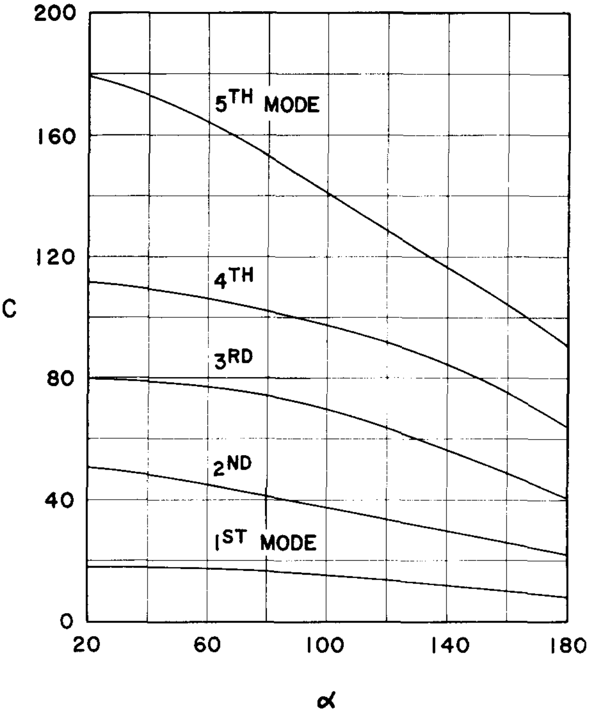

FIG. 4. Variation of C with α.

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REFERENCES

[1] J. P. DEN HARTOG, The lowest natural frequency of circular arcs. *Phil. Mag.* **5**, 400–408 (1928).
[2] E. VOLTERRA and J. D. MORELL, A note on the lowest natural frequency of elastic arcs. *J. appl. Mech.* **27**, 744–746 (1960).
[3] F. W. WALTKING, Schwingungszahlen und schwingungsformen von kreisbogentragern. *Ing.-Arch.* **5**, 429–449 (1934).
[4] L. S. D. MORLEY, The flexural vibrations of a cut thin ring. *Q. Jl. mech. appl. Math.* **11**, 491–497 (1958).
[5] R. R. ARCHER, Small vibrations of thin incomplete circular rings. *Int. J. Mech. Sci.* **1**, 45–56 (1960).
[6] H. C. NELSON, In-plane vibration of a simply supported circular ring segment. *Int. J. Mech. Sci.* **4**, 517–527 (1962).
[7] J. I. PARCEL and R. B. B. MOORMAN, *Analysis of Statically Indeterminate Structures.* John Wiley (1955).
[8] S. P. TIMOSHENKO and J. M. GERE, *Theory of Elastic Stability.* McGraw-Hill (1961).
[9] W. NOWACKI, *Dynamics of Elastic Systems.* John Wiley (1963).

APPENDIX

Case 1

$$\mathbf{D}(\theta) = [\cos\sigma_1\theta \quad \cos\sigma_2\theta \quad \cos\sigma_3\theta \quad \sin\sigma_1\theta \quad \sin\sigma_2\theta \quad \sin\sigma_3\theta]$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1^2 & -\sigma_2^2 & -\sigma_3^2 & 0 & 0 & 0 \\ \hline & & \mathbf{R} & & & \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1-\sigma_1^2 & 1-\sigma_2^2 & 1-\sigma_3^2 & 0 & 0 & 0 \\ \sin\beta & \sin\beta & \sin\beta & \sigma_1\cos\beta & \sigma_2\cos\beta & \sigma_3\cos\beta \\ -\cos\beta & -\cos\beta & -\cos\beta & \sigma_1\sin\beta & \sigma_2\sin\beta & \sigma_3\sin\beta \\ \hline & & \mathbf{R} & & & \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1-\sigma_1^2 & 1-\sigma_2^2 & 1-\sigma_3^2 & 0 & 0 & 0 \\ \cos\beta & \cos\beta & \cos\beta & -\sigma_1\sin\beta & -\sigma_2\sin\beta & -\sigma_3\sin\beta \\ \sin\beta & \sin\beta & \sin\beta & \sigma_1\cos\beta & \sigma_2\cos\beta & \sigma_3\cos\beta \\ \hline & & \mathbf{R} & & & \end{bmatrix}$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\sigma_1\alpha & \cos\sigma_2\alpha & \cos\sigma_3\alpha & \sin\sigma_1\alpha & \sin\sigma_2\alpha & \sin\sigma_3\alpha \\ -\sigma_1\sin\sigma_1\alpha & -\sigma_2\sin\sigma_2\alpha & -\sigma_3\sin\sigma_3\alpha & \sigma_1\cos\sigma_1\alpha & \sigma_2\cos\sigma_2\alpha & \sigma_3\cos\sigma_3\alpha \\ -\sigma_1^2\cos\sigma_1\alpha & -\sigma_2^2\cos\sigma_2\alpha & -\sigma_3^2\cos\sigma_3\alpha & -\sigma_1^2\sin\sigma_1\alpha & -\sigma_2^2\sin\sigma_2\alpha & -\sigma_3^2\sin\sigma_3\alpha \end{bmatrix}$$

Case 2

$$\mathbf{D}(\theta) = [\cos\sigma_1\theta \quad \cos\mu\theta\cosh\nu\theta \quad \cos\mu\theta\sinh\nu\theta \quad \sin\sigma_1\theta \quad \sin\mu\theta\sinh\nu\theta \quad \sin\mu\theta\cosh\nu\theta]$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & \sigma_1 & 0 & \mu \\ -\sigma_1^2 & \nu^2-\mu^2 & 0 & 0 & 2\mu\nu & 0 \\ \hline & & \mathbf{S} & & & \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1-\sigma_1^2 & 1+\nu^2-\mu^2 & 0 & 0 & 2\mu\nu & 0 \\ \sin\beta & \sin\beta & \nu\cos\beta & \sigma_1\cos\beta & 0 & \mu\cos\beta \\ -\cos\beta & -\cos\beta & \nu\sin\beta & \sigma_1\sin\beta & 0 & \mu\sin\beta \\ \hline & & \mathbf{S} & & & \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1-\sigma_1^2 & 1+\nu^2-\mu^2 & 0 & 0 & 2\mu\nu & 0 \\ \cos\beta & \cos\beta & -\nu\sin\beta & -\sigma_1\sin\beta & 0 & -\mu\sin\beta \\ \sin\beta & \sin\beta & \nu\cos\beta & \sigma_1\cos\beta & 0 & \mu\cos\beta \\ \hline & & & \mathbf{S} & & \end{bmatrix}$$

where

$$\mathbf{S} = \begin{bmatrix} \cos\sigma_1\alpha & \cos\mu\alpha\cosh\nu\alpha & \cos\mu\alpha\sinh\nu\alpha \\ -\sigma_1\sin\sigma_1\alpha & \begin{array}{c}-\mu\sin\mu\alpha\cosh\nu\alpha\\ +\nu\cos\mu\alpha\sinh\nu\alpha\end{array} & \begin{array}{c}-\mu\sin\mu\alpha\sinh\nu\alpha\\ +\nu\cos\mu\alpha\cosh\nu\alpha\end{array} \\ -\sigma_1^2\cos\sigma_1\alpha & \begin{array}{c}(\nu^2-\mu^2)\cos\mu\alpha\cosh\nu\alpha\\ -2\mu\nu\sin\mu\alpha\sinh\nu\alpha\end{array} & \begin{array}{c}(\nu^2-\mu^2)\cos\mu\alpha\sinh\nu\alpha\\ -2\mu\nu\sin\mu\alpha\cosh\nu\alpha\end{array} \end{matrix}$$

$$\begin{matrix} \sin\sigma_1\alpha & \sin\mu\alpha\sinh\nu\alpha & \sin\mu\alpha\cosh\nu\alpha \\ \sigma_1\cos\sigma_1\alpha & \begin{array}{c}\mu\cos\mu\alpha\sinh\nu\alpha\\ +\nu\sin\mu\alpha\cosh\nu\alpha\end{array} & \begin{array}{c}\mu\cos\mu\alpha\cosh\nu\alpha\\ +\nu\sin\mu\alpha\sinh\nu\alpha\end{array} \\ -\sigma_1^2\sin\sigma_1\alpha & \begin{array}{c}(\nu^2-\mu^2)\sin\mu\alpha\sinh\nu\alpha\\ +2\mu\nu\cos\mu\alpha\cosh\nu\alpha\end{array} & \begin{array}{c}(\nu^2-\mu^2)\sin\mu\alpha\cosh\nu\alpha\\ +2\mu\nu\cos\mu\alpha\sinh\nu\alpha\end{array} \end{bmatrix}.$$

Case 3

$$\mathbf{D}(\theta) = [\cos\sigma_1\theta \quad \cosh\sigma_2\theta \quad \cosh\sigma_3\theta \quad \sin\sigma_1\theta \quad \sinh\sigma_2\theta \quad \sinh\sigma_3\theta]$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1^2 & \sigma_2^2 & \sigma_3^2 & 0 & 0 & 0 \\ \hline & & & \mathbf{T} & & \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1-\sigma_1^2 & 1+\sigma_2^2 & 1+\sigma_3^2 & 0 & 0 & 0 \\ \sin\beta & \sin\beta & \sin\beta & \sigma_1\cos\beta & \sigma_2\cos\beta & \sigma_3\cos\beta \\ -\cos\beta & -\cos\beta & -\cos\beta & \sigma_1\sin\beta & \sigma_2\sin\beta & \sigma_3\sin\beta \\ \hline & & & \mathbf{T} & & \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1-\sigma_1^2 & 1+\sigma_2^2 & 1+\sigma_3^2 & 0 & 0 & 0 \\ \cos\beta & \cos\beta & \cos\beta & -\sigma_1\sin\beta & -\sigma_2\sin\beta & -\sigma_3\sin\beta \\ \sin\beta & \sin\beta & \sin\beta & \sigma_1\cos\beta & \sigma_2\cos\beta & \sigma_3\cos\beta \\ \hline & & & \mathbf{T} & & \end{bmatrix}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos\sigma_1\alpha & \cosh\sigma_2\alpha & \cosh\sigma_3\alpha & \sin\sigma_1\alpha & \sinh\sigma_2\alpha & \sinh\sigma_3\alpha \\ -\sigma_1\sin\sigma_1\alpha & \sigma_2\sinh\sigma_2\alpha & \sigma_3\sinh\sigma_3\alpha & \sigma_1\cos\sigma_1\alpha & \sigma_2\cosh\sigma_2\alpha & \sigma_3\cosh\sigma_3\alpha \\ -\sigma_1^2\cos\sigma_1\alpha & \sigma_2^2\cosh\sigma_2\alpha & \sigma_3^2\cosh\sigma_3\alpha & -\sigma_1^2\sin\sigma_1\alpha & \sigma_2^2\sinh\sigma_2\alpha & \sigma_3^2\sinh\sigma_3\alpha \end{bmatrix}$$

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Абстракт—Для определия собственных частот рамных кюнструкций выводятся общие динамические уравнения угловых деформаций, для круглых членов постоянного сечения. В качестве примера, даётся двухпролётная закривленная рама для иллюстрации выведенных уровнений и для указания влияния зффекта центрального угла арки на собственные частоты рамы.